DOCUMENT RESUME

ED 123 088 SE 020 744

AUTHOR	Weaver, J. F.
TITLE	Calculator-Influenced Explorations in School Mathematics: Number Sentences and Sentential Transformations I, II. Project Paper 76-1.
INSTITUTION	Wisconsin Univ., Madison. Research and Development Center for Cognitive Learning.
SPONS AGENCY	National Inst. of Education (DHEW), Washington, D.C.
REPORT NO	WRDCCL-PP-76-1
PUB DATE	Jan 76
CONTRACT	NE-C-00-3-0065
NOTE	53p.; Report from the Project on Analysis of Mathematics Instruction
EDRS PRICE	MF-$0.83 HC-$3.50 Plus Postage
DESCRIPTORS	Curriculum; Elementary Education; *Elementary School Mathematics; *Instruction; *Instructional Aids; Instructional Materials; *Mathematics Education; Number Systems; *Research; Worksheets
IDENTIFIERS	*Hand Calculators

ABSTRACT

Explorations involving the use of mini electronic calculators in connection with mathematics instruction were conducted with two fifth-grade classes, two second-grade classes, and one third-grade class. Explorations are being continued with third-grade classes (last year's second grade classes). Selected activities and related materials are described and illustrated. Limited empirical data suggest that pupils encounter no consequential problems with the mechanics of using simple four-function, algebraic-logic calculators in routine contexts, and that pupils likely will elect not to use calculators in situations where their use is unnecessary or of no particular advantage. Departing from elementary-school mathematics programs which usually emphasize binary operations and their properties in relation to distinguished sets of numbers, project explorations have moved increasingly toward content interpretations in terms of unary operations and operators, and properties associated with them. (Author/SD)

****
* Documents acquired by ERIC include many informal unpublished *
* materials not available from other sources. ERIC makes every effort *
* to obtain the best copy available. Nevertheless, items of marginal *
* reproducibility are often encountered and this affects the quality *
* of the microfiche and hardcopy reproductions ERIC makes available *
* via the ERIC Document Reproduction Service (EDRS). EDRS is not *
* responsible for the quality of the original document. Reproductions *
* supplied by EDRS are the best that can be made from the original. *
****

ED123088

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
NATIONAL INSTITUTE OF EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

Project Paper 76-1

Calculator-influenced Explorations in School Mathematics: Number Sentences and Sentential Transformations I,II

by

J. F. Weaver

Report from the Project on

ANALYSIS OF MATHEMATICS INSTRUCTION

Wisconsin Research and Development
Center for Cognitive Learning
The University of Wisconsin
Madison, Wisconsin

January 1976

Published by the Wisconsin Research and Development Center for Cognitive Learning, supported in part as a research and development center by funds from the National Institute of Education, Department of Health, Education, and Welfare. The opinions expressed herein do not necessarily reflect the position or policy of the National Institute of Education and no official endorsement by that agency should be inferred.

Center Contract No. NE-C-00-3-0065

SE 020 744

ABSTRACT

Explorations involving the use of mini electronic calculators in connection with mathematics instruction were conducted in the Stephens Elementary School (Madison, WI) with two fifth-grade classes during the 1973-74 school year, and with two second-grade classes and one third-grade class during the 1974-75 school year. During the current school year (1975-76) the explorations are being continued with the two third grade classes (last year's second grade classes). Selected activities and related materials are described and illustrated. Limited empirical data suggest that pupils encounter no consequential problems with the mechanics of using simple four-function, algebraic-logic calculators in routine contexts, and that pupils likely will elect not to use calculators in situations where their use is unnecessary or of no particular advantage.

Elementary-school mathematics programs usually emphasize binary operaoperations (+ - × ÷) and their properties in relation to distinguished sets of numbers. Project explorations, however, have moved increasingly toward content interpretations in terms of unary operations and operators, and properties associated with them.

INTRODUCTION

During the 1972-73 academic year numerous conversations with my colleague, Prof. T. A. Romberg, led to an approved long-range research direction pertaining broadly to the use of electronic calculators (particularly the "mini" or "pocket" or "hand-held" variety) in connection with school mathematics instruction. Considerable informal exploration would be necessary before profitable controlled experimentation would be feasible. Although such exploration should be somewhat diffuse, some focus for greater emphasis should be identified at the outset.

This Project Paper (76-1) is an interim report that draws upon a few of the explorations begun in the fall of 1973 and still in progress. (More complete information will be found in Technical Reports and similar documents that are in preparation.) Particular emphasis has been given to explorations associated with many of the equation types identified in Table 1 (page 3). This focus of special interest stemmed from the fact that some fruitful investigations already had been completed, or were in progress, regarding whole-number sentences (Weaver, 1971,1972,1973a; Montague, 1972; Grouws, 1972,1974; Grouws & Good, 1976; McMaster, 1976) and also the distributive idea (Weaver, 1973b; Hobbs, 1976). It was felt that the calculator very well might be used to advantage in work involving number sentences and sentential transformations, and properties of operations.

General considerations. Several things which apply across all years of the project to date should be noted at the outset.

1. I wish to reiterate that the work has involved informal exploration rather than controlled experimentation. Although some empirical data were collected and are reported, the principal intent of the project to date has not been hypothesis formulation and testing. (That will come at a later time.)

2. All explorations were conducted within the same Madison, WI location: the Glenn W. Stephens Elementary School; Donald Stern, Principal (and a most cooperative and supportive one, to say the least).

TABLE 1

Some Simple Equation Types

I. $a \nabla b = \square$	II. $a \nabla \square = c$	III. $\square \nabla b = c$
1. $a + b = \square$	5. $a + \square = c$	9. $\square + b = c$
2. $a - b = \square$	6. $a - \square = c$	10. $\square - b = c$
3. $a \times b = \square$	7. $a \times \square = c$	11. $\square \times b = c$
4. $a \div b = \square$	8. $a \div \square = c$	12. $\square \div b = c$

IV. $(a \nabla b) \# c = \square$	V. $a \nabla (b \# c) = \square$
13. $(a + b) + c = \square$	29. $a + (b + c) = \square$
14. $(a + b) - c = \square$	30. $a + (b - c) = \square$
15. $(a + b) \times c = \square$	31. $a + (b \times c) = \square$
16. $(a + b) \div c = \square$	32. $a + (b \div c) = \square$
17. $(a - b) + c = \square$	33. $a - (b + c) = \square$
18. $(a - b) - c = \square$	34. $a - (b - c) = \square$
19. $(a - b) \times c = \square$	35. $a - (b \times c) = \square$
20. $(a - b) \div c = \square$	36. $a - (b \div c) = \square$
21. $(a \times b) + c = \square$	37. $a \times (b + c) = \square$
22. $(a \times b) - c = \square$	38. $a \times (b - c) = \square$
23. $(a \times b) \times c = \square$	39. $a \times (b \times c) = \square$
24. $(a \times b) \div c = \square$	40. $a \times (b \div c) = \square$
25. $(a \div b) + c = \square$	41. $a \div (b + c) = \square$
26. $(a \div b) - c = \square$	42. $a \div (b - c) = \square$
27. $(a \div b) \times c = \square$	43. $a \div (b \times c) = \square$
28. $(a \div b) \div c = \square$	44. $a \div (b \div c) = \square$

3. The extent of exploratory work for each participating class was decidedly limited. During each of the school years it involved between 14 and 18 once-a-week 25-minute sessions.

4. The exploratory work was independent of ongoing mathematics programs. No particular attempt was made to articulate a class's calculator explorations with its day-to-day school mathematics work. Care was taken, however, to avoid any possible interference between calculator explorations and ongoing mathematics programs.

5. During the first two years the classroom teachers elected not to have pupils use the calculators in connection with any of their day-to-day mathematics work. During the current year, however, teachers are opting to have pupils use the calculators from time to time in connection with the ongoing mathematics programs.

6. For all of the exploratory sessions, I was the teacher--but had the benefit of some degree of assistance, as indicated in subsequent sections of this report.

7. The pupil-to-calculator ratio was one-to-one: within a given class, each pupil used--and was responsible for--the same calculator at each weekly session.

8. All calculators (with two minor exceptions during the current year) were of the kind pictured in Figure 1 (page 5): four-function, algebraic-logic (without operational hierarchy) calculators, with a switch permitting abbreviated keystroking for multiplication or division by a constant.

The remainder of this report is organized in three sections: one for each of the three school years during which explorations have been conducted. The material included in each section is in no way definitive. It may not even be representative! It was selected to illustrate those aspects of exploration that were of principal interest from my personal point of view and that will be given a major continuing emphasis as the project progresses into the future.

DATAMATH Electronic Calculator (TI-2500)

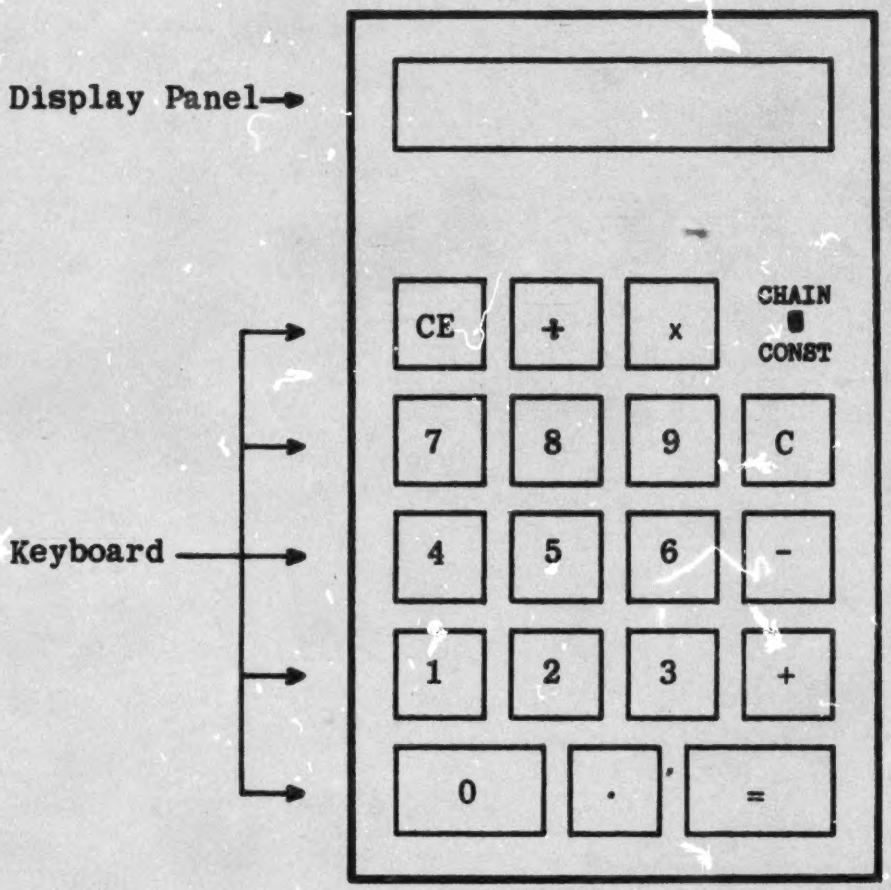


Figure 1. The calculator type used by pupils

THE 1973-74 SCHOOL YEAR

These calculator-influenced explorations were conducted with the two fifth-grade classes. Scheduling considerations made it preferable to meet these pupils in four groups.

At each group's weekly session I was assisted by Prof. Robert Bechtel and Ms. Mary Jane McMaster. Prof. Bechtel, on leave from the Purdue University Calumet Campus, also contributed significantly in many other ways in planning and implementing the explorations. Ms. McMaster was a Ph.D. candidate in mathematics education at The University of Wisconsin-Madison. The pupils' regular classroom teachers were not involved in the instructional work of the calculator sessions.

Many of this year's activities and experiences involved things that were facilitated by use of a calculator but did not necessitate the use of a calculator. Page 7 is a simple illustration of this. Finding all possible counting-number solutions for the given equation is facilitated by using a calculator and, especially, its "constant" multiplication function. For the TI-2500's used by the pupils, the constant operator was associated with the second factor of a multiplication expression (box C of Table 2, page 8) rather than with the first factor (box F of Table 2). Pupils had to be aware of this in order to use to advantage this "constant function" feature of the calculator.

Other activities and experiences illustrated for pupils the need to record systematically at times the data or information generated by using a calculator. Consider two somewhat different instances of this:

Page 9. When the calculator is used as a "Factor Finder" to see if 5, for instance, is a factor of 168, it is sufficient simply to note that the display is not a whole number when 168 is divided by 5; consequently 5 is a "Dud" (not a factor of 168). There is no need to record the display per se. But when 8, for instance, is tested it is not sufficient simply to note that the quotient is a whole number and thus 8 is a "Keeper" (a factor of 168). The quotient of 21 also is a factor of 168. By recording it, the extent of the testing procedure is reduced.

Page 10. This is a slightly abbreviated version of sheets used in connection with the process of dividing. (Our Z-notation for this process is,

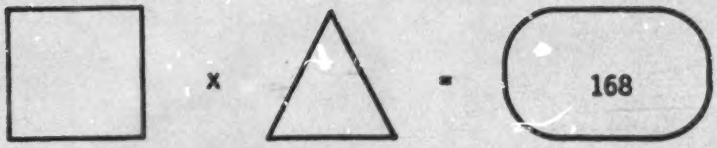
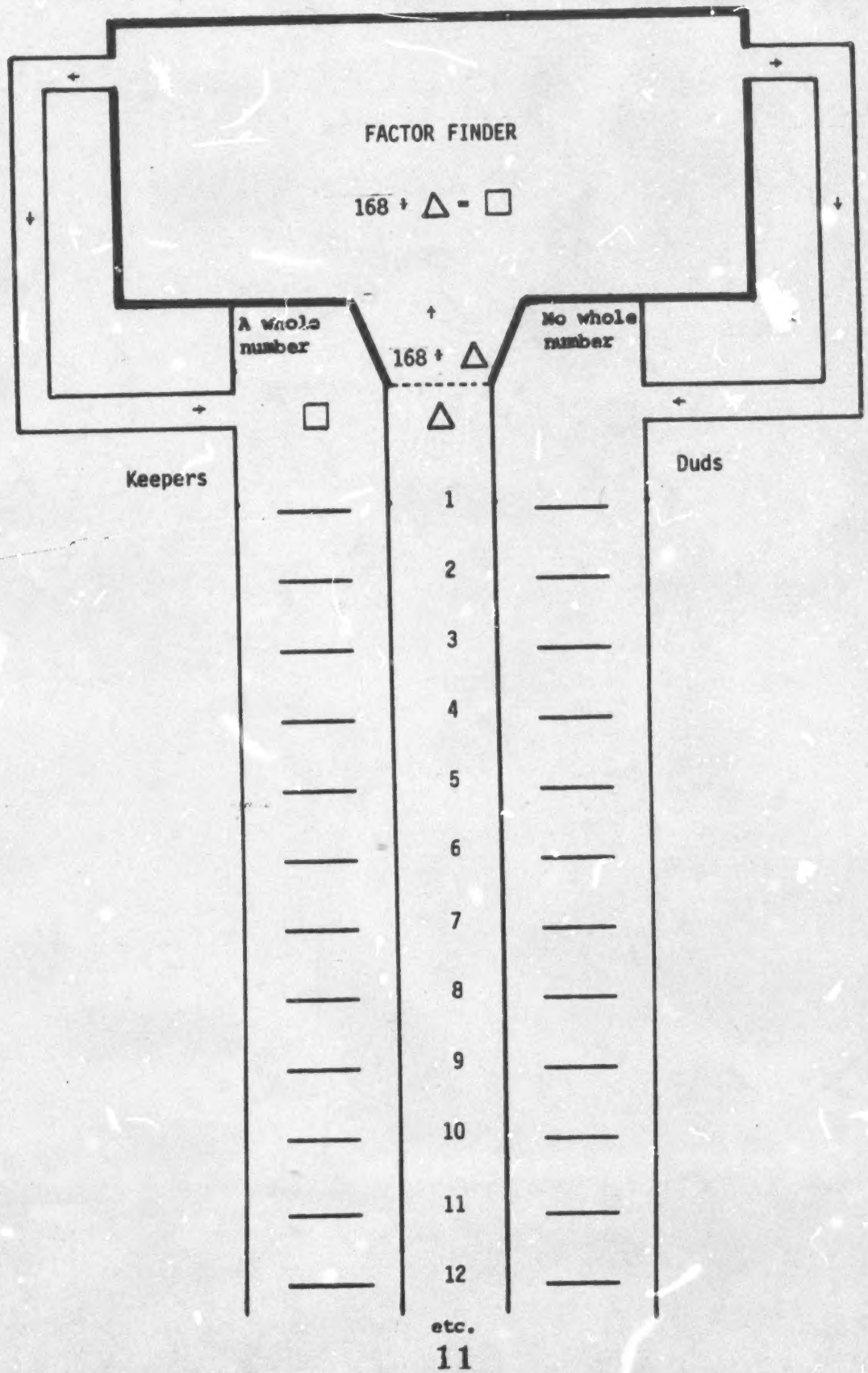

x
=
168

TABLE 2

Illustrative Calculator "Constant" Functions

	Constant post-operator		Constant pre-operator	
A	18 + 3 = 21 12 = 15 21 = 24	or	18 + 3 = 21 12 = 30 21 = 39	E
B	18 - 3 = 15 12 = 9 21 = 18			
C	18 x 3 = 54 12 = 36 21 = 63	or	18 x 3 = 54 12 = 216 21 = 378	F
D	18 ÷ 3 = 6 12 = 4 21 = 7			

Note. Underlining is used to indicate the operator held constant by the calculator in each instance.


FACTOR FINDER
168 ÷ △ = □
A whole number
168 ÷ △
No whole number
Keepers
Duds
1
2
3
4
5
6
7
8
9
10
11
12
etc.

DIVIDING

a Z b → (q, r). 4985 Z 79 → (_____ q, _____ r)

1. _____ a ÷ _____ b → Display Q

So, q = _____

2. _____ q × _____ b → Display q × b

3. _____ a - _____ q × b → Display r

I believe, unique.) Although the product $q \times b$ in step 2 is but an intermediate result, it is important to record it so that it will not be forgotten, since it is re-entered via the calculator keyboard in step 3.

Note that in step 1 when 4085 is divided by 79, the display Q will be 63.101265 (or 63.101266 on some calculators), which is truncated mentally to give q, or 63. The recording of Q is desirable rather than necessary.

(It may be of interest to note that on calculators like the TI-2500, if the keystroke sequence "4085 [÷] 79 [÷] 10000000 [×] 10000000 [=]" had been executed, the display would read 63. That is to say, the truncation of Q (63.101265) to q (63) is effected by the calculator rather than mentally by the user. This procedure was not used with pupils, however; for them it would be nothing more than a trick of some sort, since the rationale of the calculation would be beyond their ken.)

I now turn to several things that were more within the domain of the project's principal emphases.

Chaining. On an algebraic-logic calculator with no operational hierarchy, such as the TI-2500, "chaining" pertains to keystroke sequences of the form "a [▽] b [#] c [=]," where [▽] and [#] are operational keys ([+] or [−] or [×] or [÷]) and a and b and c are strings of digit keys for numbers. An instance of such chaining would be "539 [+] 185 [−] 681 [=]." The mechanics of chaining are simple--so deceptively simple, in fact, that at times conceptually erroneous results are generated. To clarify this, I shall backtrack to a portion of Table 1 (page 3).

Consider first several class I equations such as "4 - 7 = □," "9 ÷ 2 = □" and "3 ÷ 8 = □." If work is being restricted to the domain or universe of whole numbers, none of these three equations has a solution. A calculator such as the TI-2500 is unaware that any such restriction has been placed upon our work. Consequently, for the given equations the calculator displays "- 3," "4.5" and "0.375" (or ".375') respectively as solutions. The pupil must sense that these displays are not whole numbers (or names for whole numbers, if you insist) so that in fact the equations have no whole number solutions.

Now consider an equation such as "4 - 7 + 8 = □." Within the domain or universe of whole numbers, this equation has no solution. But a calculator will blithely "chain" and display a whole number solution of 5, disregarding that "4 - 7" or "- 3" is not a whole number and adding 8 to it.

Pupils must learn how to detect intermediate displayed results during the chaining process.

Sheets like page 13 were found to be helpful lead-ins to chaining and some of its features. The progression from keystroke sequence A to B to C reduced pupils' perplexity when they observed that pushing the [-] key in step 5 of keystroke sequence C generated a display of 724, which is the sum ([+]) of 539 and 185.

Sheets like page 14 were used periodically in connection with chaining and other keystroking sequences as a means of recording information step by step. Pupils were encouraged to "plan ahead" by writing the keying sequences first, and then executing them and recording displays.

(In subsequent years the sheets like page 14 were replaced by the Calculator Record Sheet which could be used for the same purpose in either of several ways as illustrated by pages 15 and 16.)

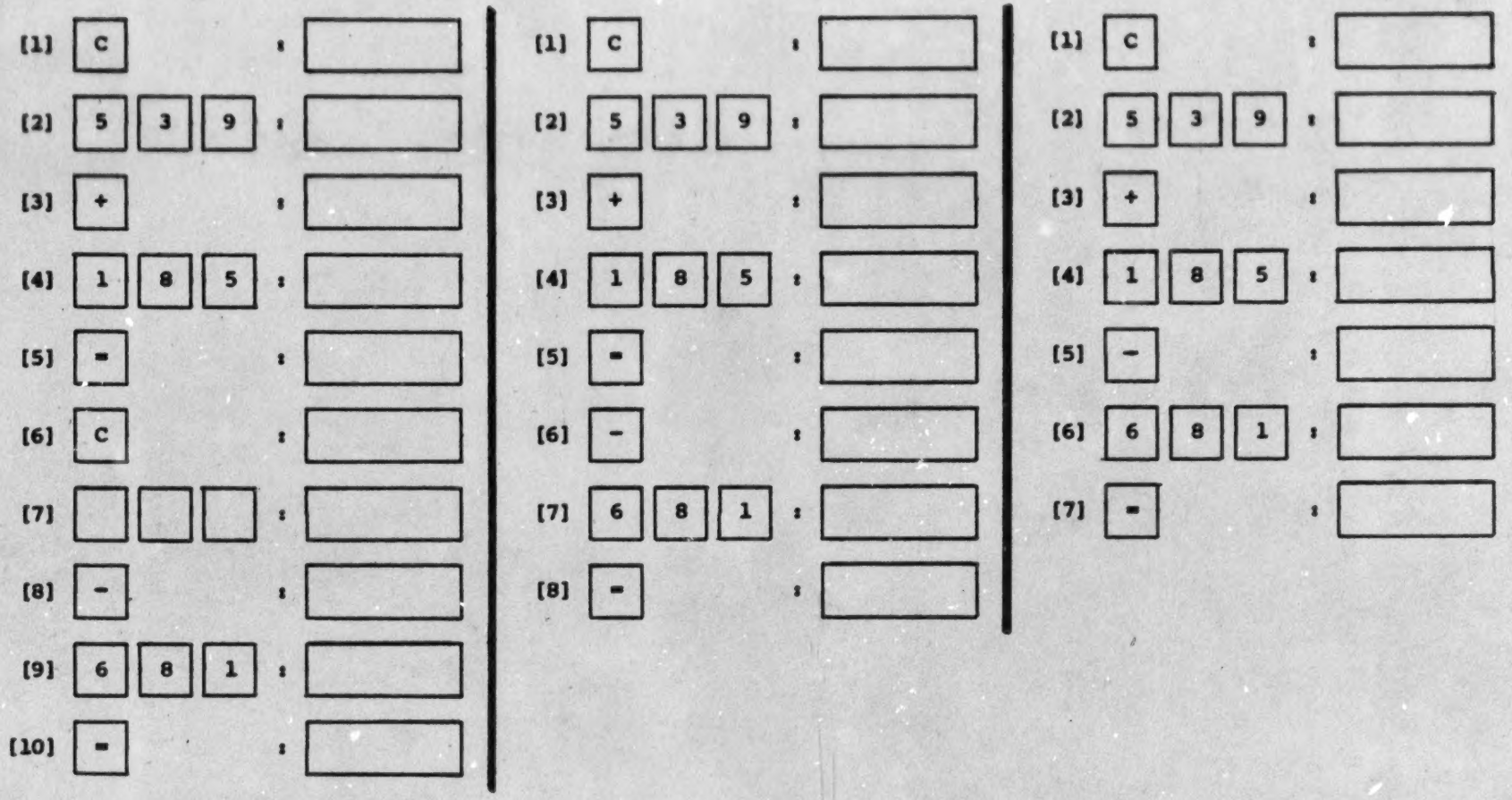
Note that in step 6 of page 14, a minus sign appeared at the far left of the display--indicating that in effect the negative integer "- 681" had been entered into the calculator. This posed some problems of interpretation and was more of a nuisance than anything else! This minus sign at such a point has been suppressed from the display in many other calculators, including the TI-2500-II.

Even when a minus sign was desired or necessary in a display, it was unfortunate that in the TI-2500 the sign appeared at the extreme left of the display--being overlooked too many times by some pupils. The "floating" minus sign of the TI-2500-II and certain other calculators is a better arrangement for elementary-school pupils.

Finally, in connection with chaining, it was very necessary to make clear that this process treated examples as though they had parentheses as in equation types 13-28 of Table 1. Equation types 29-44 are a quite different thing and do not convey the same meaning as 13-28. Apart from illustrations that were used to make this distinction, very little exploration was done with instances of equation types 29-44.

Doing-and-Undoing. This idea was explored progressively in two stages, both of which emphasized unary rather than binary operations. This distinction is characterized in Table 3 (page 17)--for the reader, not the pupil!--and the right- or post-operator interpretation was the one used

Step	Key(s) Pushed	Display
1st	C	0.
2nd	5 3 9	539.
3rd	+	539.
4th	1 8 5	185.
5th	-	724.
6th	6 8 1	- 681.
7th	=	43.
8th		
9th		
10th		
11th		
12th		
13th		
14th		
15th		


Keystroke sequence A
[1] C :
[2] 5 3 9 :
[3] + :
[4] 1 8 5 :
[5] - :
[6] C :
[7] :
[8] - :
[9] 6 8 1 :
[10] - :
Keystroke sequence B
[1] C :
[2] 5 3 9 :
[3] + :
[4] 1 8 5 :
[5] - :
[6] - :
[7] 6 8 1 :
[8] - :
Keystroke sequence C
[1] C :
[2] 5 3 9 :
[3] + :
[4] 1 8 5 :
[5] - :
[6] 6 8 1 :
[7] - :

CALCULATOR RECORD SHEET

Step	Keys		Display
1		539	539.
2	+		539.
3		185	185.
4	-		724.
5		681	- 681.
6	=		43.
7			
8			
9			
10			
11			
12			

CALCULATOR RECORD SHEET

Step	Keys		Display
1		539	539.
2	+	185	185.
3	-	681	- 681.
4	=		43.
5			
6			
7			
8			
9			
10			
11			
12			

TABLE 3

Binary and Unary Conceptualizations of Certain Operations Involving Whole Numbers

Binary conceptualization			Unary conceptualization					
			Left- or pre-operator interpretation			Right- or post-operator interpretation		
START with this	DO this	GET this	START with this	DO this	GET this	START with this	DO this	GET this
(a,b)	+	a + b or c	b	<u>a +</u>	a + b or c	a	<u>+ b</u>	a + b or c
(a,b)	-	a - b or c				a	<u>- b</u>	a - b or c
(a,b)	×	a × b or c	b	<u>a ×</u>	a × b or c	a	<u>× b</u>	a × b or c
(a,b)	÷	a ÷ b or c				a	<u>÷ b</u>	a ÷ b or c
(8,2)	+	8 + 2 or 10	2	<u>8 +</u>	8 + 2 or 10	8	<u>+ 2</u>	8 + 2 or 10
(8,2)	-	8 - 2 or 6				8	<u>- 2</u>	8 - 2 or 6
(8,2)	×	8 × 2 or 16	2	<u>8 ×</u>	8 × 2 or 16	8	<u>× 2</u>	8 × 2 or 16
(8,2)	÷	8 ÷ 2 or 4				8	<u>÷ 2</u>	8 ÷ 2 or 4

in the explorations.

Working within the whole-number domain, a problem was started as on page 19. Pupils used calculators to determine the number for the GET box. Then pupils conjectured how to complete the sheet as on page 20, and calculators were used to verify or deny the conjectures. Physical-world analogs were used as needed to clarify the conceptualizations.

In this first stage, then, pupils were in effect establishing the following relationships:

$$a \underline{+ b} = c. \quad \text{So} \quad c \underline{- b} = a.$$
$$a \underline{- b} = c. \quad \text{So} \quad c \underline{+ b} = a.$$
$$a \underline{\times b} = c. \quad \text{So} \quad c \underline{\div b} = a.$$
$$a \underline{\div b} = c. \quad \text{So} \quad c \underline{\times b} = a$$

In the second stage pupils used sheets like pages 21 and 22 to abbreviate the work of the first stage, leading to these relationships:

$$a \underline{- b} \underline{+ b} = a$$
$$a \underline{+ b} \underline{- b} = a$$
$$a \underline{\times b} \underline{\div b} = a$$
$$a \underline{\div b} \underline{\times b} = a$$

These two stages dealt with certain of the properties identified in Table 4 (page 23), which then were used in connection with the next set of explorations.

Related Number Sentences. This takes us to the heart of this year's explorations.

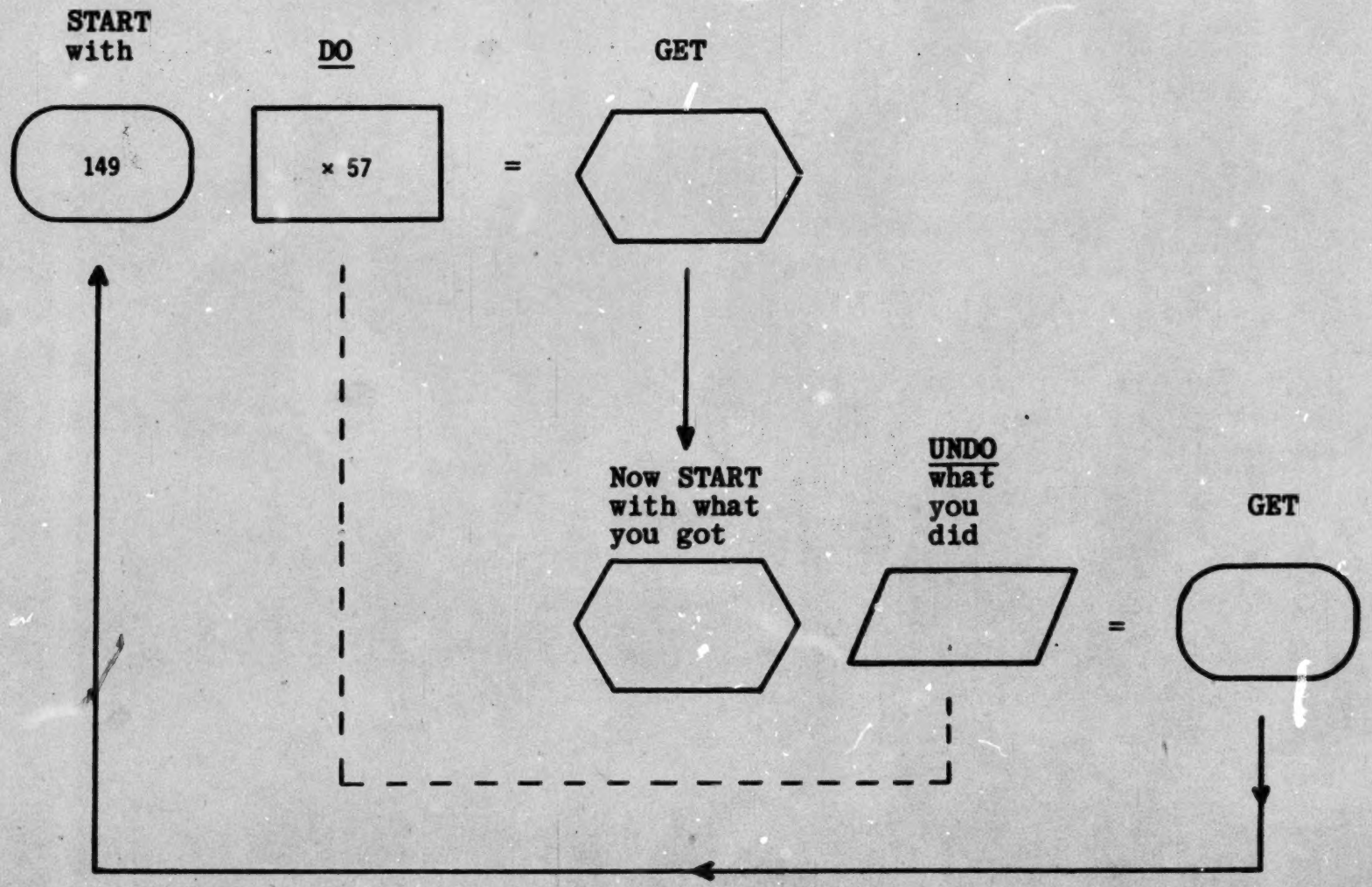
It is commonplace for school mathematics programs to have pupils work with sets of related "basic number facts" such as these:

$5 + 9 = 14$	$14 - 9 = 5$	$8 \times 3 = 24$	$24 \div 3 = 8$
$9 + 5 = 14$	$14 - 5 = 9$	$3 \times 8 = 24$	$24 \div 8 = 3$

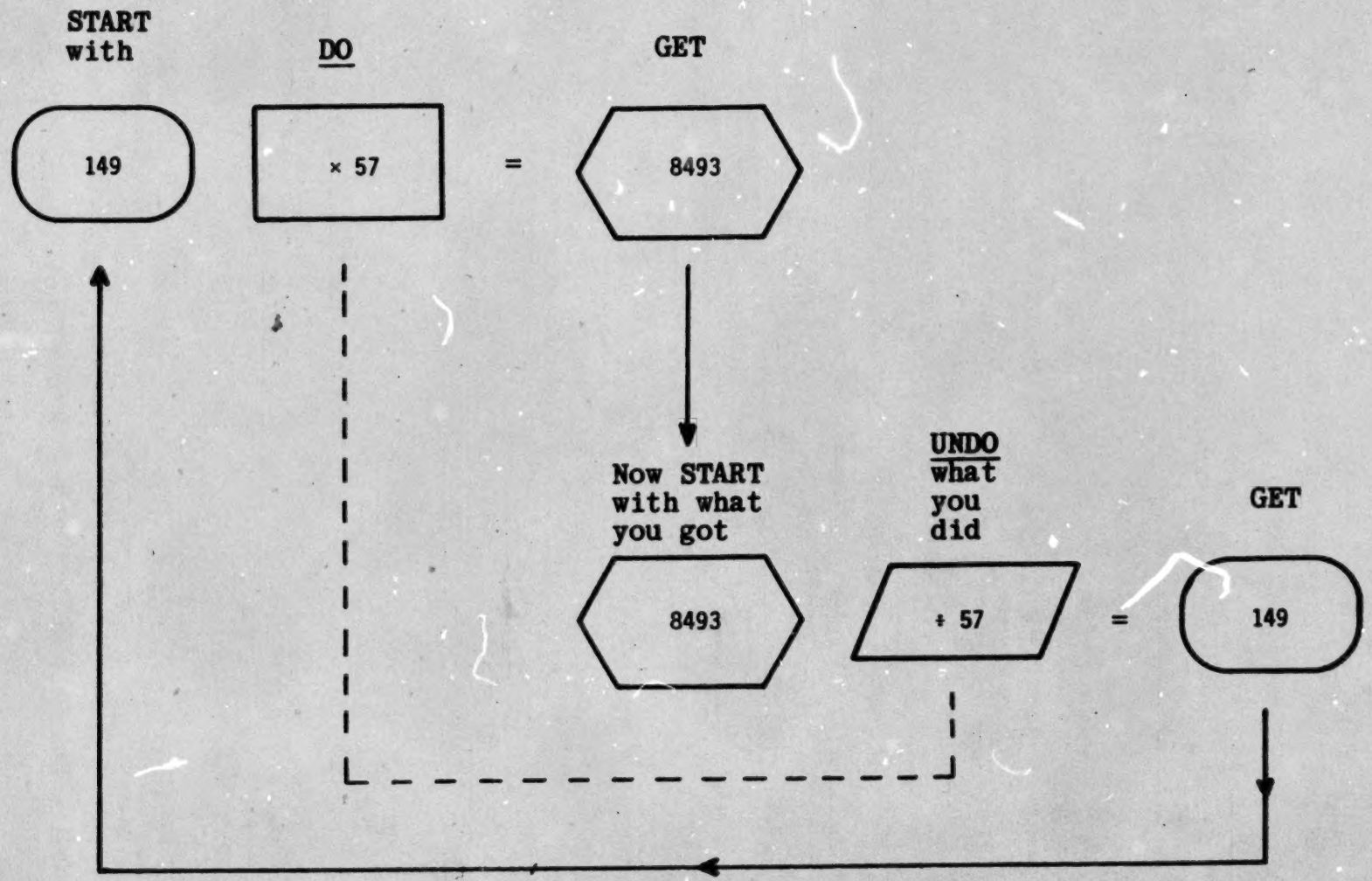
Rarely is this same relatedness extended to larger whole numbers or to other sets of numbers. In the belief that it would be advantageous to do so, such an extension was made for the whole-number domain.

Pupils were given an open sentence such as "$578 + 693 = \square$." This would be solved using calculators and the resulting closed sentence "$578 + 693 = 1271$" would be recorded in box W of a sheet like page 24. Closed sentences would be conjectured and written for boxes X, Y and Z.

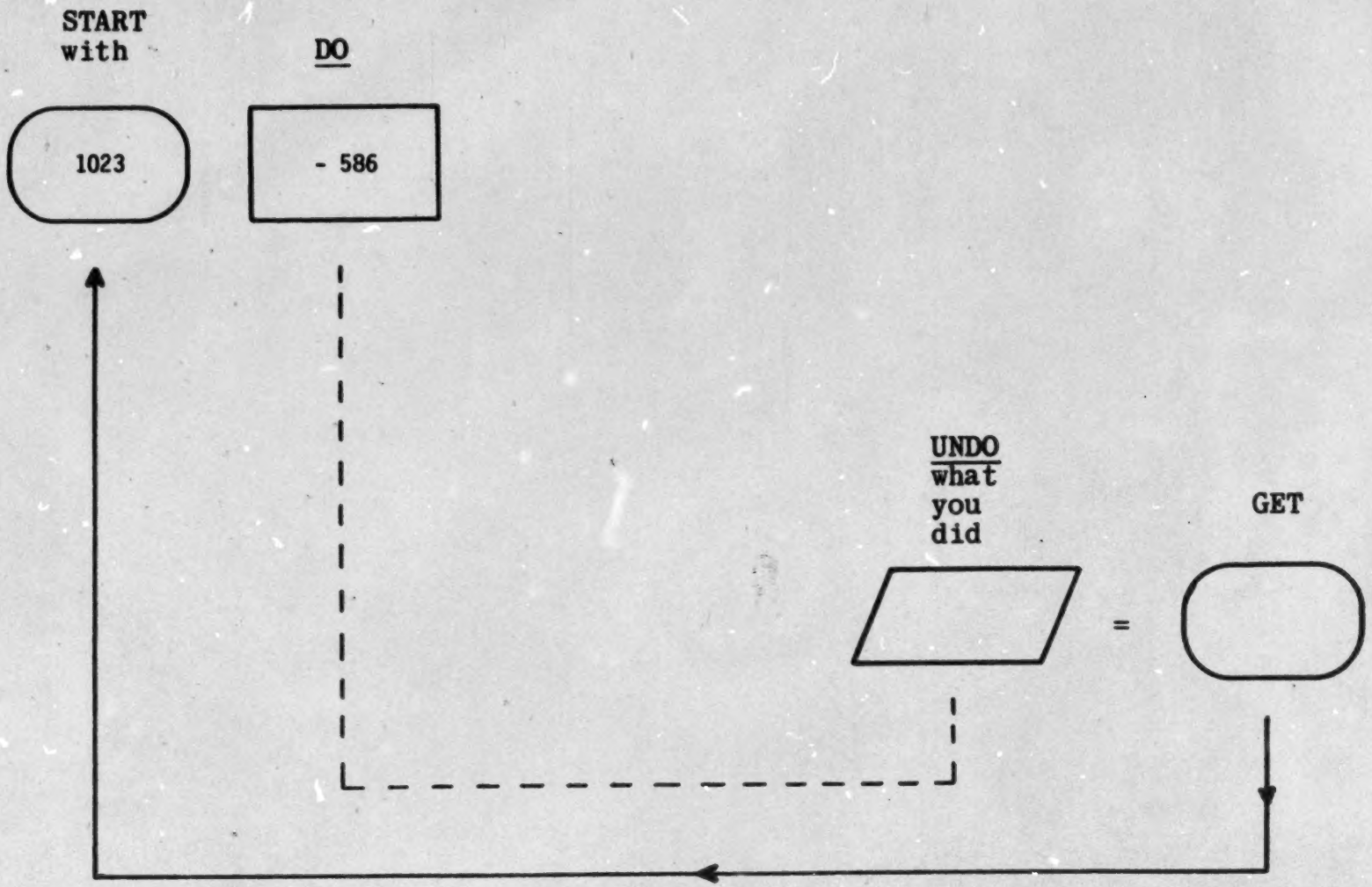
The DOING-UNDOING Idea



The DOING-UNDOING Idea



The DOING-UNDOING Idea



The DOING-UNDOING Idea

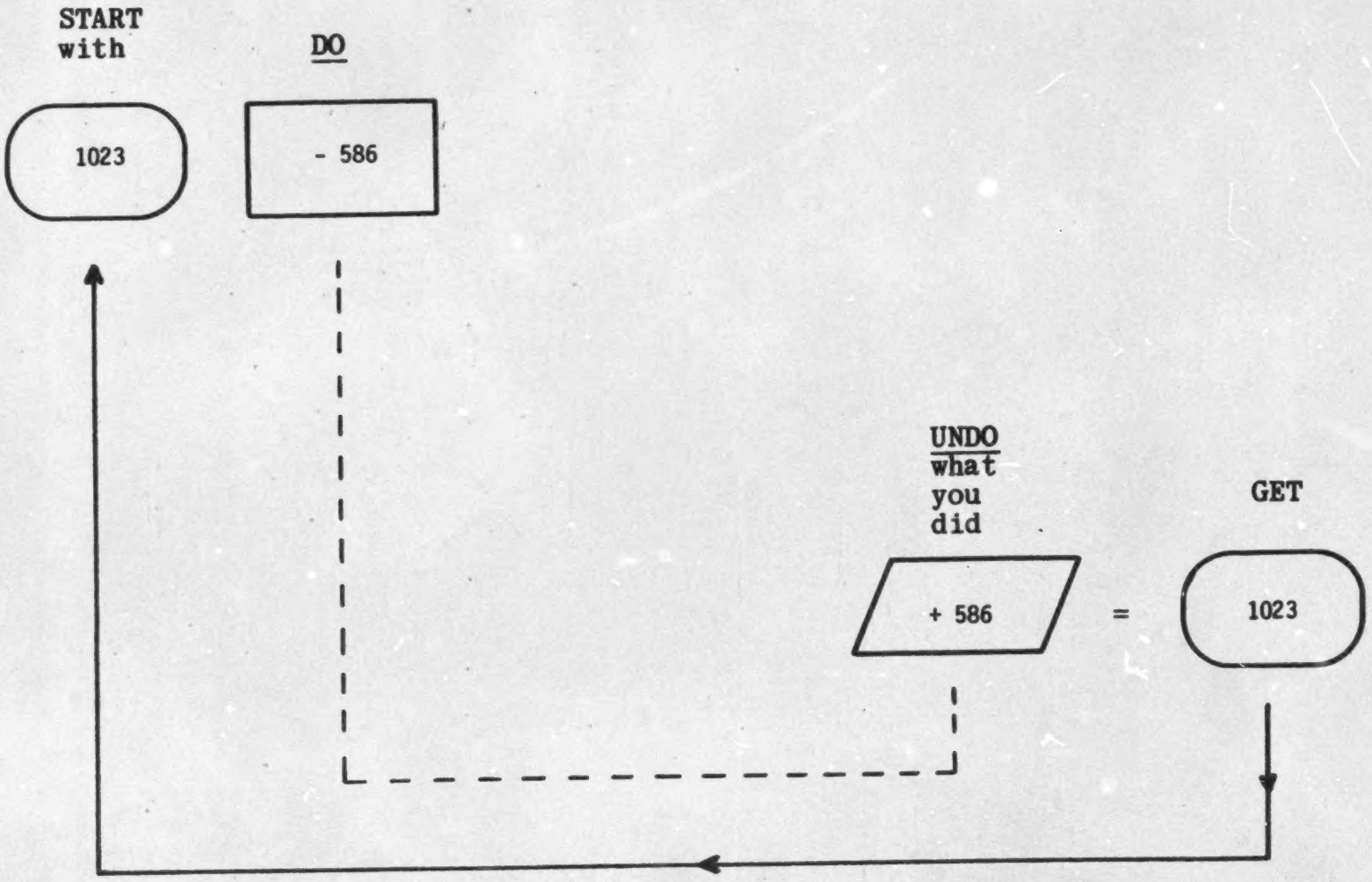


TABLE 4

Some Properties Associated with Certain Unary Operations

within the Domain and Range of Whole Numbers

1. $a + b = b + a$	13. $a \times b = b \times a$
2. $a + 0 = a$	14. $a \times 1 = a$
3. $a - 0 = a$	15. $a \div 1 = a$
4. $a - a = 0$	*16. $a \div a = 1$
	17. $a \times 0 = 0$
5. $a + b - b = a$	*18. $a \times b \div b = a$
* 6. $a - b + b = a$	*19. $a \div b \times b = a$
7. $a + b + c = a + c + b$	20. $a \times b \times c = a \times c \times b$
* 8. $a - b - c = a - c - b$	*21. $a \div b \div c = a \div c \div b$
* 9. $a + b - c = a - c + b$	*22. $a \times b \div c = a \div c \times b$
*10. $a - b + c = a + c - b$	*23. $a \div b \times c = a \times c \div b$
§11. IF $a + b = c$, THEN $c - b = a$	§24. IF $a \times b = c$, THEN $c \div b = a$
§12. IF $a - b = c$, THEN $c + b = a$	§25. IF $a \div b = c$, THEN $c \times b = a$

Note.--The preceding properties clearly are not wholly independent, but each is included in the listing for purposes of emphasis.

* Certain restriction(s) must be placed upon the operator(s).

§ This is an elided form of the statement, "If it is true that . . ., then it also is true that . . ."

Related Number Sentences

	DOING	UNDOING	
W	578 + 693 = 1271	1271 - 693 = 578	X
Y	693 + 578 = 1271	1271 - 578 = 693	Z

Note.--It is important for pupils to realize that the column labels, DOING and UNDOING, may be interchanged at will. The sentences in boxes X and Z may be interpreted as DOING, and those in boxes W and Y as respective UNDOING.

using suitable properties, linked with physical-world analogs as needed. The validity of the conjectured sentences in boxes X, Y and Z then would be checked using calculators.

Page 26 illustrates a later step in this progression of work. Given the open sentence "4183 + ☐ = 47," for instance, it was written in box W and related open sentences were written in boxes X, Y and Z. The solution then was derived from the open sentence in box Y (in this instance) using a calculator; all sentences then were rewritten as closed sentences (set B), and the validity of each was tested using the calculator.

Finally, the preceding work was expanded to include the symmetric forms of sentences, as illustrated by the incomplete page 27.

More than a trivial number of pupils found it difficult to distinguish between (1) transforming one sentence to another using the commutative property associated with addition or multiplication and (2) transforming one sentence to another using the symmetric property of the equality relation. It was not uncommon for pupils to change sentences of the form "a + b = c" to sentences of the form "c = b + a" (and similarly for multiplication), and also to change sentences of the form "a - b = c" to sentences of the form "c = b - a" (and similarly for division). Difficulties such as these prompted less emphasis than originally intended upon the symmetric forms of related sentences.

Some data pertaining to pupils' performance were gathered:

1. Prior to any work with related number sentences a 12-item subtest was administered. Item content is summarized in Table 5 (page 28) which does not indicate, however, the subtest format. For each item two questions were asked: Is [a] the same number as [n] ? Is [b] the same number as [n] ? Without computing or calculating, pupils were to respond YES or NO or UNC (uncertain; can't be sure without finding the numbers) to each of the two questions for each of the 12 items. This same subtest, but with items sequenced in a different order, was administered again after the explorations with related number sentences were completed.

The proportion of mathematically correct question responses increased from 55% to 64%. The proportion of "Plus" item responses (both questions answered correctly) increased from 39% to 49%; the proportion of "Minus" item responses (neither question answered correctly) decreased from 28% to 19%. The proportion of "Zero" item responses (exactly one of two questions

Related Number Sentences (A)

	DOING	UNDOING	
W	$4183 \div \square = 47$	$47 \times \square = 4183$	X
Y	$4183 \div 47 = \square$	$\square \times 47 = 4183$	Z

Related Number Sentences (B)

	DOING	UNDOING	
W	$4183 \div 89 = 47$	$47 \times 89 = 4183$	X
Y	$4183 \div 47 = 89$	$89 \times 47 = 4183$	Z

Related Number Sentences (A)

	DOING	UNDOING	
W	□ + 289 = 1004		X
Y			Z

Related Number Sentences (A')

	SYMMETRIC FORMS		
W'	1004 = □ + 289		X'
Y'			Z'

TABLE 5

Sentences for Subtest 1-A

Item No.	Stem	Responses	
1	[n] + 568 = 713	713 + 568 = [a]	[b] = 713 - 568
2	987 ÷ [n] = 47	987 ÷ 47 = [a]	[b] = 47 × 987
3	[n] - 596 = 738	738 + 596 = [a]	[b] = 738 - 596
4	48 × [n] = 912	912 × 48 = [a]	[b] = 912 ÷ 48
5	624 - [n] = 485	485 + 624 = [a]	[b] = 624 - 485
6	[n] × 37 = 851	851 ÷ 37 = [a]	[b] = 37 × 851
7	379 + [n] = 804	804 - 379 = [a]	[b] = 379 + 804
8	[n] ÷ 392 = 56	392 ÷ 56 = [a]	[b] = 392 × 56
9	[n] × 43 = 817	817 × 43 = [a]	817 = 43 × [b]
10	[n] - 527 = 486	486 - [a] = 527	486 = 527 - [b]
11	754 ÷ [n] = 29	[a] = 29 ÷ 754	[b] ÷ 754 = 29
12	563 + [n] = 912	912 = [a] + 563	912 + [b] = 563

answered correctly) was essentially unchanged: 33% and 32%.

2. Subtest 1-B consisted of the eight open sentences listed in Table 6. The pupils were required to use calculators to solve each open sentence, recording the calculator display in each instance.

TABLE 6

Items for Subtest 1-B

1. $7013 - \boxed{a} = 6145$	5. $1647 = \boxed{e} - 3829$
2. $9675 = \boxed{b} \times 387$	6. $12 \times \boxed{f} = 8676$
3. $469 = 9849 \div \boxed{c}$	7. $\boxed{g} \div 522 = 29$
4. $\boxed{d} + 1594 = 6380$	8. $4512 = 2673 + \boxed{h}$

Each pupil response (apart from relatively few "omits") was scored as conceptually correct or incorrect, disregarding any more or less technical errors that obviously could be attributed to miskeying the calculator, misreading the calculator display, etc. On this basis, 72.5% of the pupils gave conceptually correct responses to at least six of the eight items (75% criterion level). Technical errors were identified with only 3% of the pupils' responses, distributed proportionately between conceptually correct and conceptually incorrect responses.

In conclusion. All in all, Mr. Stern and other school personnel were gratified with the intent, nature, conduct, and tangible and intangible outcomes of the year's explorations. It was agreed without reservation that calculator-influenced explorations at one or more lower grade levels would be appropriate and desirable during the next school year.

THE 1974-75 SCHOOL YEAR

These calculator-influenced explorations were conducted with the two second-grade classes and the one fifth-grade class. At each class's weekly session I was assisted by the pupils' regular classroom teacher: Mrs. Elizabeth Livermore, or Mrs. Elizabeth Vilhauer, or Mrs. Nancy Love.

It was decided to provide essentially the same explorations for all classes, subject to modification if necessitated by the year's difference in age and background between the second and third grade classes. (Such modification was not necessary, however.)

Also, since it was anticipated that the work with the second grade classes could be continued when those pupils were in third grade (as has turned out to be the case), it was decided to focus especially on content associated with the addition and subtraction of whole numbers. This included solving open sentences of types 1 and 2, Table 1 (page 3); chaining and the solution of sentence types 13, 14, 17 and 18 (Table 3); the doing-undoing idea and other selected properties; and limited work with sets of related sentences. Often activities and materials very similar to those used during the preceding year were tried at these lower grade levels. In other instances, markedly different things were used.

For example, pages 31 and 32--and similar ones using 2- and 3-place numbers--were used from the outset to make explicit the idea of unary operators (although that expression per se was not used with pupils). Relevant properties such as 1-4 were explored within the "arrow notation" context. Calculators were used to solve open sentences, to verify conjectured properties, etc.

Relationships were established between the sentence forms

$$a \xrightarrow{+b} \square \quad \text{and} \quad a + b = \square,$$

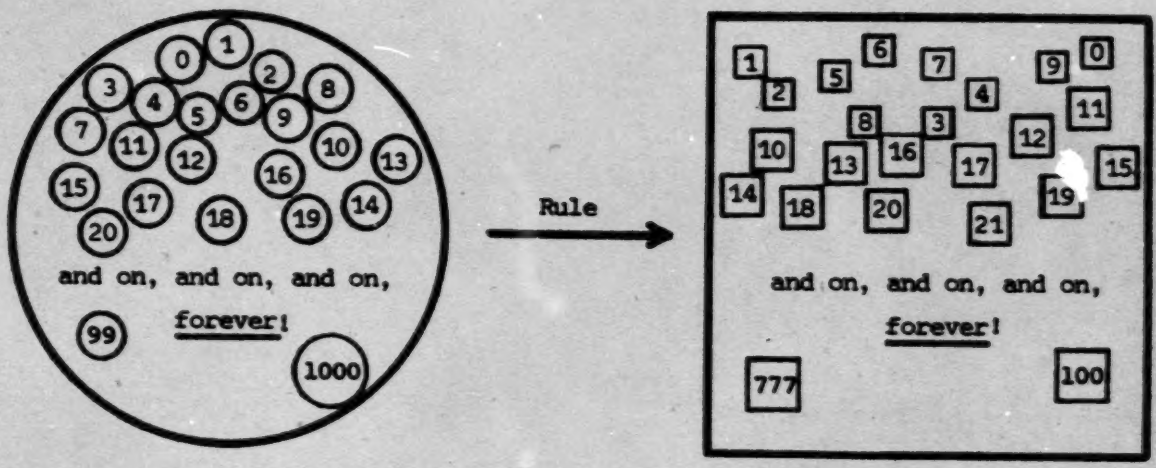
and between the setence forms

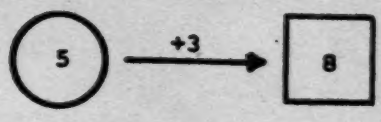
$$a \xrightarrow{-b} \square \quad \text{and} \quad a - b = \square.$$

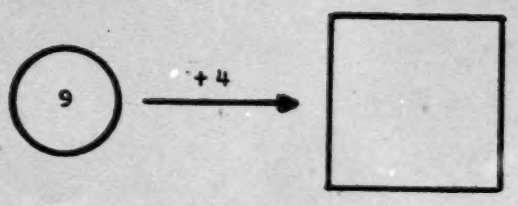
As likely was evident from the design of page 32, it was emphasized that for whole numbers a and b, "subtraction sentences" had no whole-number solutions for b greater than a. Considerable emphasis also was given to the distinction between expressions of the form "a - b" vs. "b - a," and to their counterparts in arrow notation. Apart from the special case of a = b,

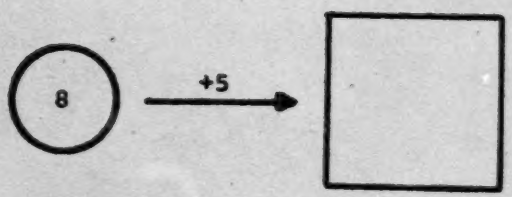
PICKING NUMBER PARTNERS

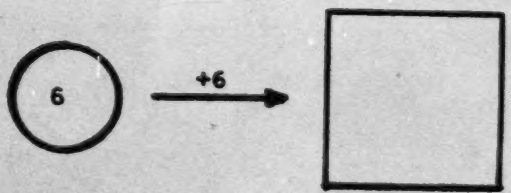
A

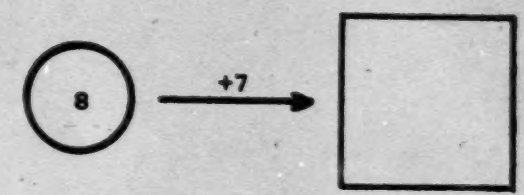


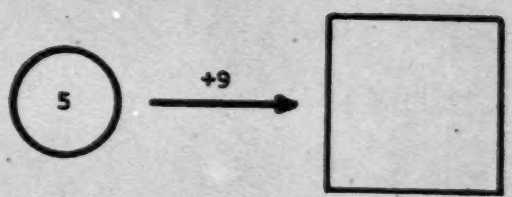


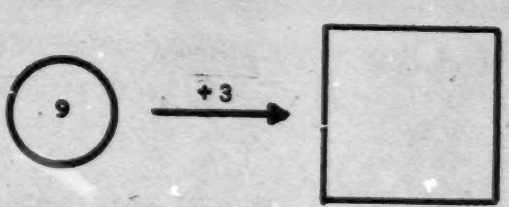






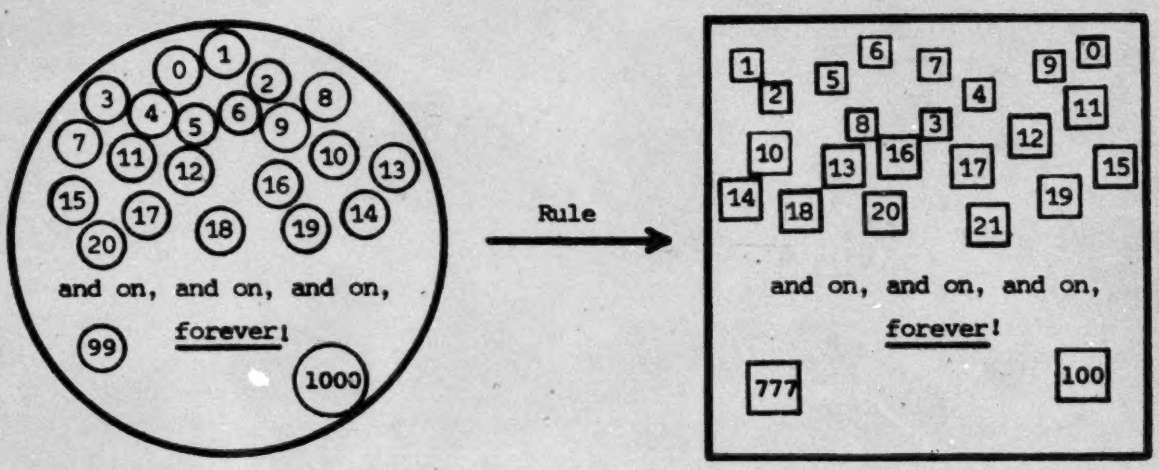


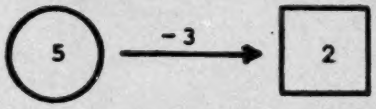


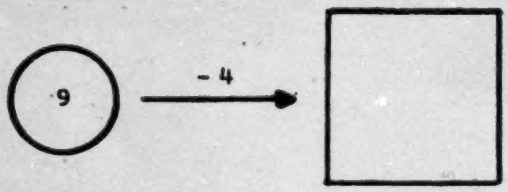


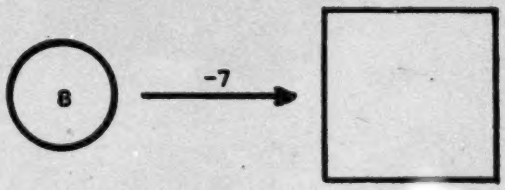
PICKING NUMBER PARTNERS

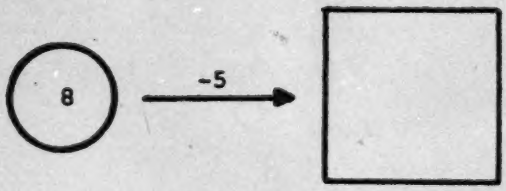
B

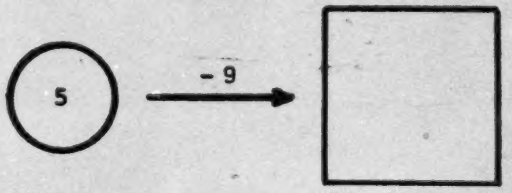


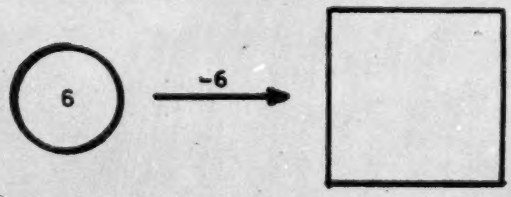


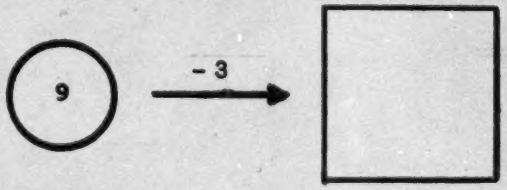












calculators can be used effectively in stressing that $a - b \neq b - a$.

Chaining. This was introduced with page 34 and similar sheets with 2- and 3-place numbers. Parentheses were eliminated after the second session of this work. All exercises were written in the more or less standard "equation" notation rather than arrow notation; however, oral discussion of sentences emphasized a unary-operator interpretation.

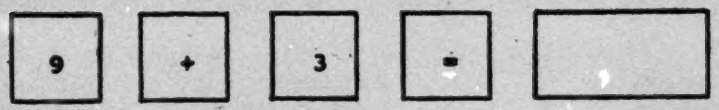
Doing-Undoing. This was introduced with page 35 and similar sheets with 2- and 3-place numbers. Sheets like pages 19-22 also were used, leading to conventional equation-notation exemplars involving the doing-undoing idea. Again, oral discussion of sentences emphasized unary-operator interpretations.

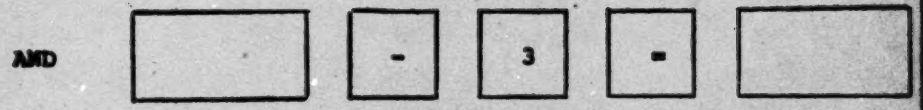
Related Number Sentences. This work was restricted to the kinds of activities and experiences discussed previously in connection with page 24. Explorations were not extended to include the kinds of things identified in connection with pages 26 and 27.

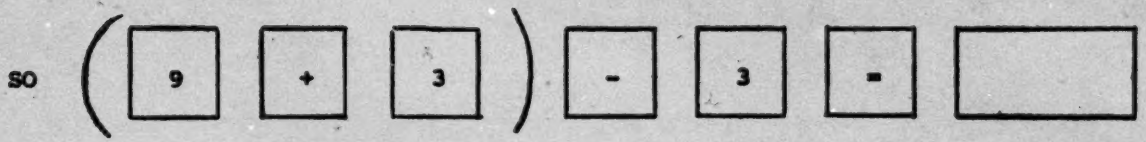
Miscellaneous. Pages 36-42 illustrate one of the more appealing activities used at periodic intervals during the course of the explorations. Transparent plastic strips of two different colors and various lengths were placed by pupils on their Counting Charts in sundry configurations, some of which are shown. Pupils conjectured which of the two sets of numbers delineated by the plastic strips had the greater or lesser sum--or the same sum--and tested their conjectures with the calculators. For pages such as 38-41 the conjectures sometimes included and sometimes excluded the number common to both sets. Numerous variations of this Counting Chart activity were explored.

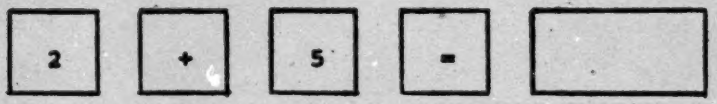
Empirical Evidence. As was true the preceding year, the gathering of empirical data was not given high priority in connection with the explorations. However, two subtests were administered to pupils toward the end of the school year. In each instance no performance differences of any consequence were noted between the two second-grade classes or between the second- and third-grade classes. Consequently, data presented here were pooled across all classes and pupils.

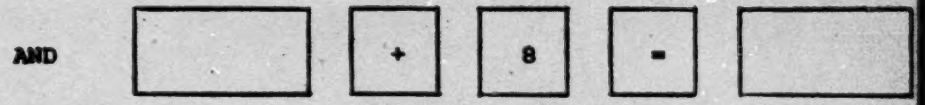
1. Subtest 2-A was designed to measure directly pupils' facility in using calculators for routine solutions to open sentences (addition and subtraction). Four forms of the five-item subtest were prepared and distributed randomly among the pupils of each class. Pupils were required to

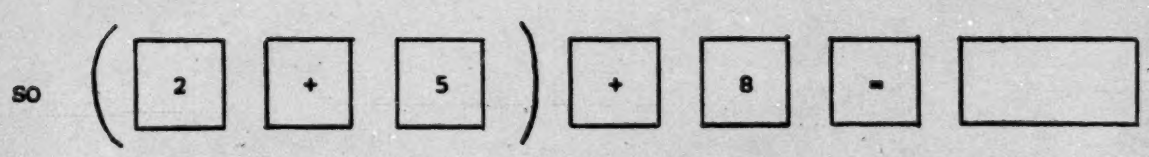

9
+
3
=

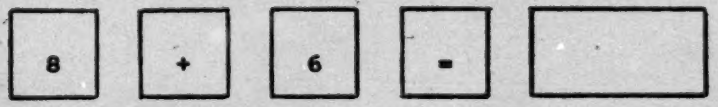

AND
-
3
=

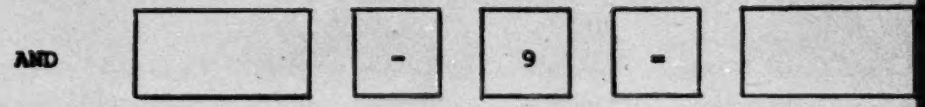

SO
9
+
3
-
3
=

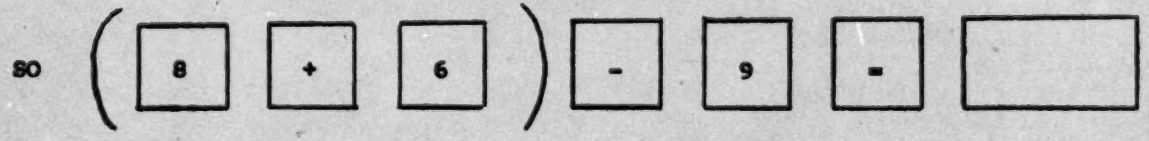

2
+
5
=


AND
+
8
=


SO
2
+
5
+
8
=

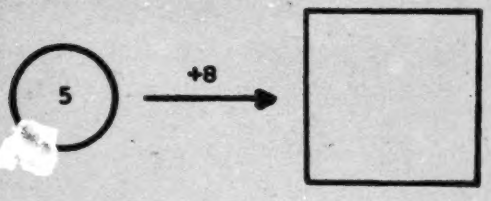

8
+
6
=


AND
-
9
=

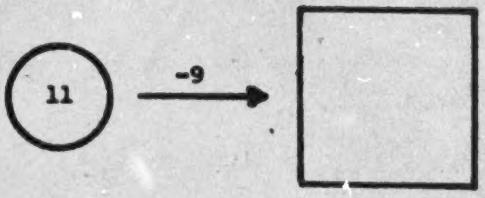

SO
8
+
6
-
9
=

[1]

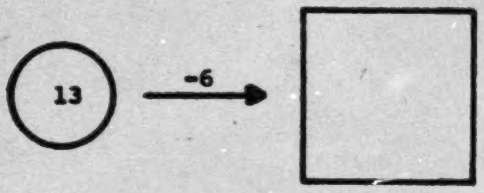
A


5
+8

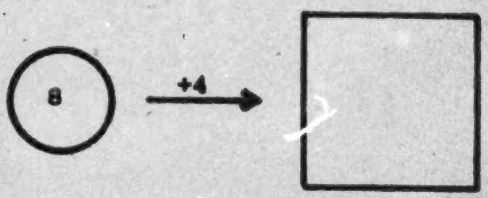
B


11
-9

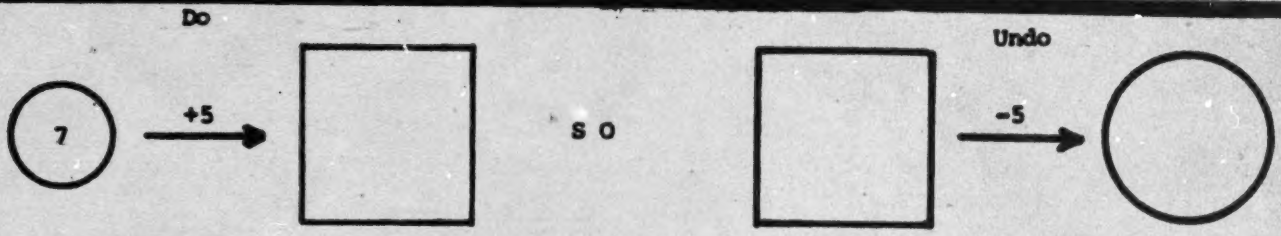
C


13
-6

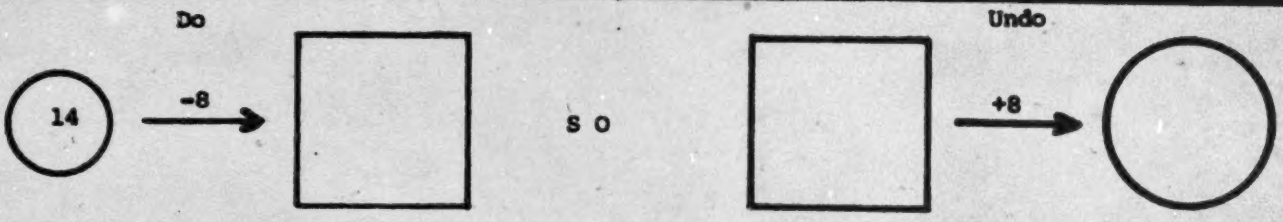
D


8
+4

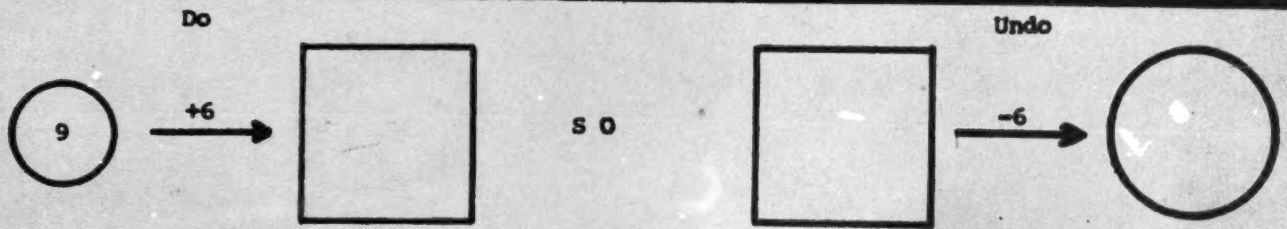
E


Do
7
+5
S O
Undo
-5

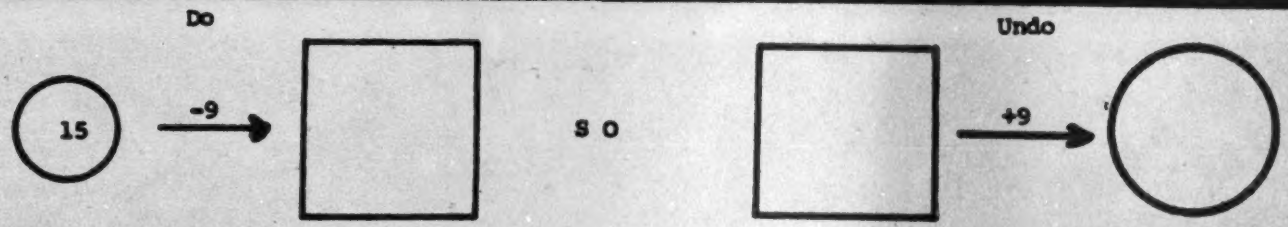
F


Do
14
-8
S O
Undo
+8

G

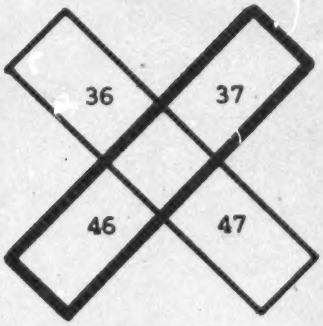

Do
9
+6
S O
Undo
-6

H


Do
15
-9
S O
Undo
+9

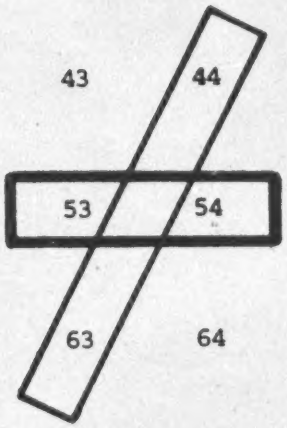
COUNTING CHART

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100



COUNTING CHART

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100



COUNTING CHART

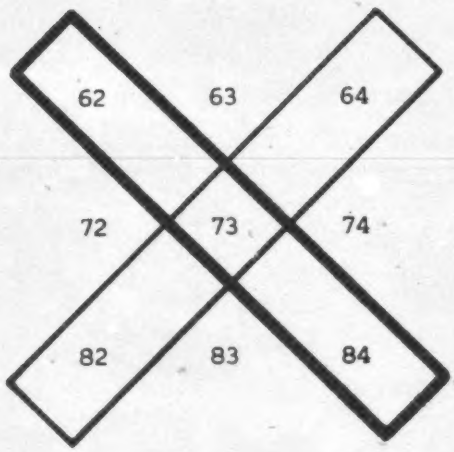
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

COUNTING CHART

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

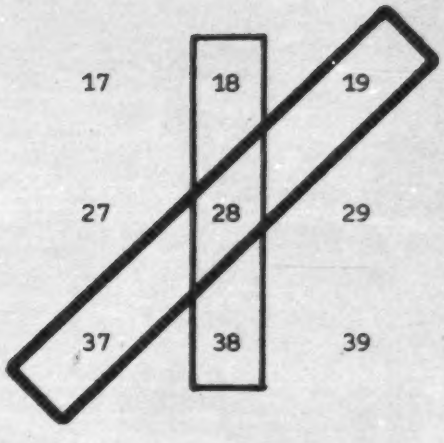
COUNTING CHART

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100



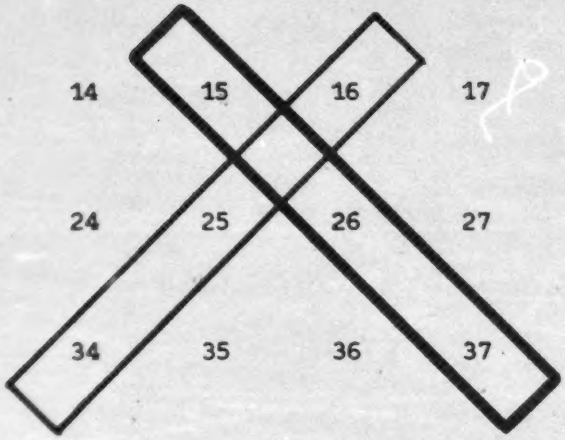
COUNTING CHART

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100



COUNTING CHART

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100



use calculators to find solutions, and space was provided on each test form for pupils to write their calculator displays (solutions). The open sentences used for subtest 2-A are listed in Table 7 (page 44).

Each pupil completed all five items of the subtest form received; no "omits" were observed. Several pupils completed a subtest within 90 seconds of working time, and only one pupil needed as much as five minutes in which to complete a subtest.

Each pupil response was scored either "correct" or "incorrect." Ninety percent of all pupil responses were correct, and 94% of the pupils responded correctly to at least four of the five subtest items (80% criterion level). Fifty-eight percent of the pupils reached a 100% criterion level.

2. Items for subtest 2-B are listed in Table 8 (page 45). Use of the calculator was optional for these items which were administered immediately following the completion of subtest 2-A, with no indication given that the subtest 2-B items were exemplars of doing-undoing or that they differed in any way from subtest 2-A items. For each item the pupil indicated two things: the solution for the open sentence, and whether or not the calculator was used in getting that solution. Spot checks of pupils at work gave some, but not full, verification of calculator use and nonuse.

It should be noted that items 6 and 7 of each subtest form were of a kind that pupils had experienced during the class explorations; but items 8, 9 and 10 were of a kind that pupils had not experienced during those explorations.

a. Across all five items, 78% of the pupil responses were correct. Twenty percent of the total pupil responses involved use of a calculator.

b. For items 6 and 7 together, 91% of the pupil responses were correct. Twenty-four percent of the total pupil responses involved use of a calculator.

c. For items 8, 9 and 10 together, 70% of the pupil responses were correct. Seventeen percent of the total pupil responses involved use of a calculator.

In conclusion. There were no reservations about continuing the explorations with the second-grade pupils during the next school year when they would be in third grade. Lack of time on my part was the only reason for not continuing explorations with the third-grade pupils when they moved into fourth grade.

TABLE 7

Items for Subtest 2-A

Item No.	Form A	Form B	Form C	Form D
1	26 + 58 = ☐	71 - 45 = ☐	57 + 26 = ☐	71 - 54 = ☐
2	1402 - 693 = ☐	894 + 308 = ☐	1401 - 693 = ☐	308 + 894 = ☐
3	374 + 85 + 249 = ☐	652 - 27 - 419 = ☐	249 + 58 + 374 = ☐	652 - 491 - 72 = ☐
4	547 + 39 - 85 = ☐	39 + 574 - 85 = ☐	607 - 283 + 311 = ☐	311 - 248 + 607 = ☐
5	311 - 287 + 603 = ☐	603 - 287 + 131 = ☐	93 + 547 - 85 = ☐	574 + 93 - 58 = ☐

TABLE 8

Items for Subtest 2-B

Item No.	Form A	Form B	Form C	Form D
6	156 + 78 - 78 = ☐	204 - 93 + 93 = ☐	165 + 87 - 87 = ☐	202 - 39 + 39 = ☐
7	201 - 39 + 39 = ☐	158 + 67 - 67 = ☐	203 - 49 + 49 = ☐	185 + 76 - 76 = ☐
8	☐ + 185 - 185 = 164	☐ - 146 + 146 = 181	☐ - 153 + 153 = 172	☐ + 146 - 146 = 158
9	302 + ☐ - 194 = 302	351 - 89 + ☐ = 351	432 + 68 - ☐ = 432	401 - ☐ + 126 = 401
10	413 + 67 - ☐ = 413	402 - ☐ + 167 = 402	307 + ☐ - 149 = 307	312 - 57 + ☐ = 312

THE 1975-76 SCHOOL YEAR

These calculator-influenced explorations are being conducted with the two third-grade classes (last year's second graders who are continuing their explorations). At each class's weekly session I am assisted by the pupils' regular classroom teacher: Mrs. Nancy Love or Mrs. Jane Scrivner.

It is premature to say much about the ongoing explorations. Pages 47-50 will indicate that work with unary operators is being continued and extended.

Page 48 and ones similar to it have been very valuable in using an approximation-correction approach to a difficult type of open sentence. These are followed with page 49 and similar ones; inverse operators (doing-undoing) are used to provide efficient, direct solutions for the same type of open sentence.

Page 50 introduces another type of open sentence (examples 5-8), to be approached first using guess-and-test procedures.

Our TI-2500 calculators do <u>not</u> have a "constant function" feature for addition and subtraction operators. This is somewhat unfortunate since a constant post-operator function (Table 2, page 8) would be distinctly advantageous with the guess-and-test work of page 48 and others like it.

#

	START with this	DO this	GET this
1.	432	+ 869	
2.	1023	- 475	

3.	734	+ 968	
4.	2146	- 1857	
5.	869	+ 432	
6.	3415	- 1987	
7.	1702	- 968	

8. 289 → (+ 1857) → ☐

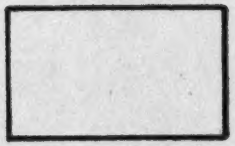
9. 1301 → (- 869) → ☐

E

+ 4 5 9 → 1 5 4 2

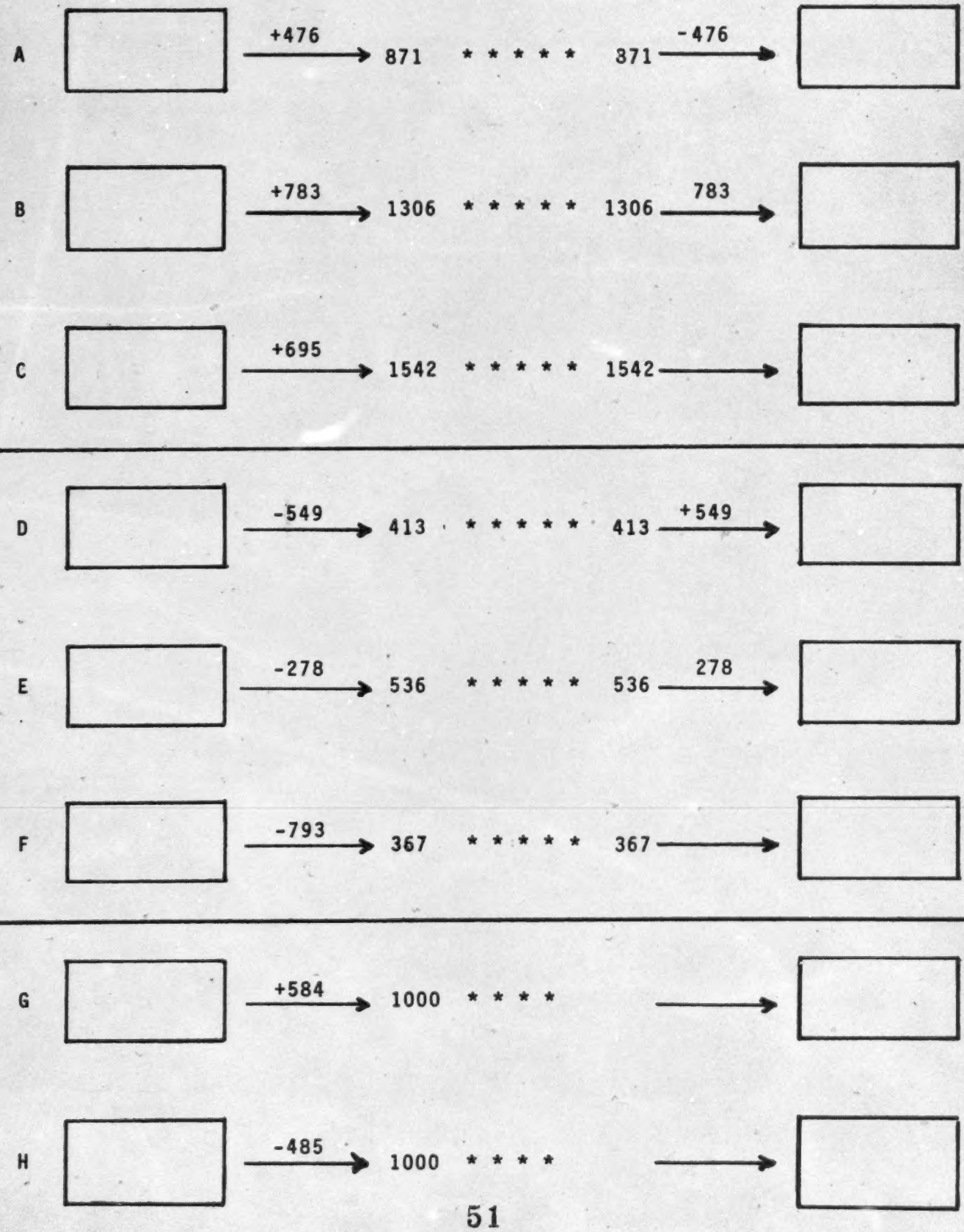
GUESS and TEST

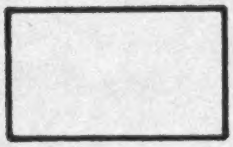
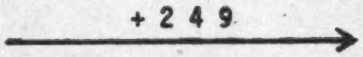
F



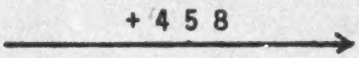
- 7 3 9 → 4 6 7

GUESS and TEST

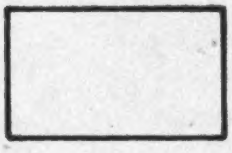
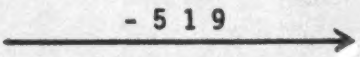

A
+476
871
* * * * *
371
-476
B
+783
1306
* * * * *
1306
783
C
+695
1542
* * * * *
1542
D
-549
413
* * * * *
413
+549
E
-278
536
* * * * *
536
278
F
-793
367
* * * * *
367
G
+584
1000
* * * *
H
-485
1000
* * * *

1.  

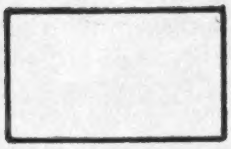
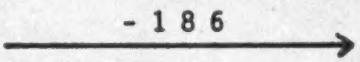
9 3 6

2. 

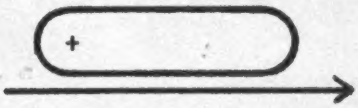
7 8 3

3.  

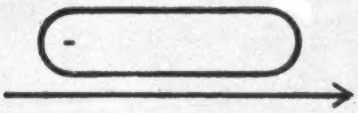
4 2 3

4.  

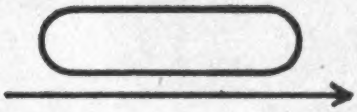
6 3 5

5. 5 3 4 

9 2 0

6. 8 0 5 

1 9 8

7. 4 9 6 
9 9 2

REFERENCES

Grouws, D. A. Differential performance of third-grade children in solving open sentences of four types. (Doctoral dissertation, University of Wisconsin, 1971). Dissertation Abstracts International, 1972, 32, 3860A.

Also: Open sentences: Some instructional considerations from research. Arithmetic Teacher, 1972, 19, 595-599.

and: Solution methods used in solving open addition and subtraction sentences. Arithmetic Teacher, 1974, 21, 255-261.

Grouws, D. A., & Good, T. L. Factors associated with third and fourth grade children's solving performance on multiplication and division sentences. Journal for Research in Mathematics Education, 1976, 7, in press.

Hobbs, C. E. An analysis of fifth-grade students' performance when solving selected open distributive sentences. (Doctoral dissertation, University of Wisconsin, 1975). Dissertation Abstracts International, in press.

McMaster, M. J. Differential performance of fourth- through sixth-grade students in solving open multiplication and division sentences. (Doctoral dissertation, University of Wisconsin, 1975). Dissertation Abstracts International, in press.

Montague, M. A. Use of matrix sampling procedures with selected examinee and item populations to assess achievement in mathematics. (Doctoral dissertation, University of Wisconsin, 1971). Dissertation Abstracts International, 1972, 32, 5475A.

Weaver, J. F. Some factors associated with pupils' performance levels on simple open addition and subtraction sentences. Arithmetic Teacher, 1971, 18, 513-519.

Also: The ability of first-, second- and third-grade pupils to identify open addition and subtraction sentences for which no solution exists within the set of whole numbers. School Science and Mathematics, 1972, 72, 679-691.

and: The symmetric property of the equality relation and young children's ability to solve open addition and subtraction sentences. Journal for Research in Mathematics Education, 1973, 4, 46-56. (a)

Weaver, J. F. Pupil performance on examples involving selected variations of the distributive idea. Arithmetic Teacher, 1973, 20, 697-704. (b)